

RESEARCH STATEMENT

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My research during the Ph.D. program mainly focuses on the Nash Equilibrium Problem and its generalization. The (generalized) Nash equilibrium problem is a kind of games to find strategies for a group of players such that each player's objective function is optimized, for given other players' strategies. Besides, my interest lies in the applications of optimization and equilibrium problems with a data-driven background.

1. THE NASH EQUILIBRIUM PROBLEMS

Let N be the number of players. Suppose the i th player's strategy is the variable $x_i \in \mathbb{R}^{n_i}$ (the n_i -dimensional Euclidean space over the real field \mathbb{R}). Denote the vector of all players' strategies as $\mathbf{x} := (x_1, \dots, x_N)$. The total dimension of all players' strategies is $n := n_1 + \dots + n_N$. For convenience, we use x_{-i} to denote the subvector of all players' strategies except the i th one, and $[N] = \{1, \dots, N\}$.

The Nash Equilibrium Problem of Polynomials (NEPP) is to find $\mathbf{x} \in \mathbb{R}^n$ such that for each $i \in N$, the i th player aims to find x_i minimizing the following optimization problem

$$(1.1) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ s.t. & g_{i,j}(x_i) \geq 0 \ (j \in \mathcal{I}_i), \\ & g_{i,j}(x_i) = 0 \ (j \in \mathcal{E}_i). \end{cases}$$

Where the f_i and $g_{i,j}$ are polynomial functions, and $\mathcal{I}_i, \mathcal{E}_i$ are disjoint labeling sets. Such a solution \mathbf{x} to the NEPP is called a Nash Equilibrium (NE).

Suppose $\mathbf{x} = (x_1, \dots, x_N)$ is a NE. Under some regularity conditions at x_i , there exist Lagrange multipliers $\lambda_{i,j}$ such that the Karush-Kuhn-Tucker (KKT) condition of the i th player's optimization hold for each $i \in [N]$. When the constraining tuple is nonsingular, the linear independent constraint qualification condition holds, and Lagrange multipliers $\lambda_{i,j}$ can be expressed as polynomials in \mathbf{x} , i.e., there exists polynomials $\lambda_{i,j}(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ such that $\lambda_{i,j}(\mathbf{x}) = \lambda_{i,j}$ for all (x_i, x_{-i}, λ_i) satisfying the KKT condition.

Assume for each player, the constraining tuple is nonsingular. Denote $m_i := |\mathcal{E}_i \cup \mathcal{I}_i|$. If \mathbf{x} is a NE, then it satisfies

$$(1.2) \quad \begin{cases} \nabla_{x_i} f_i(\mathbf{x}) - \sum_{j=1}^{m_i} \lambda_{i,j}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x_i) = 0 \ (i \in [N]), \\ g_{i,j}(x_i) = 0 \ (j \in \mathcal{E}_i, i \in [N]), \\ \lambda_{i,j}(\mathbf{x}) g_{i,j}(x_i) = 0 \ (j \in \mathcal{I}_i, i \in [N]), \\ g_{i,j}(x_i) \geq 0 \ (j \in \mathcal{I}_i, i \in [N]), \\ \lambda_{i,j}(\mathbf{x}) \geq 0 \ (j \in \mathcal{I}_i, i \in [N]). \end{cases}$$

The points satisfying (1.2) are called KKT points. Denote \mathcal{G} the set of KKT points. Consider the following optimization problem:

$$(1.3) \quad \begin{cases} \min_{\mathbf{x}} & [\mathbf{x}]_1^T \cdot \Theta \cdot [\mathbf{x}]_1 \\ s.t. & \mathbf{x} \in \mathcal{G}. \end{cases}$$

Here Θ is a generic positive definite matrix. If \mathbf{x} is a NE, then it is a feasible point of (1.3). If (1.3) is infeasible, then the NEP has no NE. Otherwise, let $u := (u_i, u_{-i})$ be a minimizer of (1.3). Consider the optimization problem for each $i \in [N]$:

$$(1.4) \quad \begin{cases} \omega_i := \min & f_i(x_i, u_{-i}) - f_i(u_i, u_{-i}) \\ s.t. & g_{i,j}(x_i) = 0 \ (j \in \mathcal{E}_i), \\ & g_{i,j}(x_i) \geq 0 \ (j \in \mathcal{I}_i). \end{cases}$$

If all the optimal values $\omega_i \geq 0$, then u is a Nash Equilibrium. However, if one of them is negative, then u is not a NE. Suppose u is not a NE. For each i such that $\omega_i < 0$, let U_i be the set of minimizers, and \mathcal{K}_i be a set containing U_i , then consider the following optimization:

$$(1.5) \quad \begin{cases} \min_{\mathbf{x}} & [\mathbf{x}]_1^T \cdot \Theta \cdot [\mathbf{x}]_1 \\ s.t. & \mathbf{x} \in \mathcal{G}, \\ & f_i(v, x_{-i}) - f_i(x_i, x_{-i}) \geq 0 (v \in \mathcal{K}_i, i \in [N]). \end{cases}$$

The u is not a feasible point of (1.5), since $f_i(v, u_{-i}) - f_i(u_i, u_{-i}) < 0$ for all $v \in \mathcal{K}_i$ and $i \in [N]$. On the other hand, each NE is still feasible for (1.5). Therefore, we can solve (1.5) for a minimizer. If it is verified to be a NE, then we are done. If it is not, we update \mathcal{K}_i by $\mathcal{K}_i \cup U_i$. Repeating this procedure, we are able to find a Nash equilibrium. We propose an Algorithm to find a NE based on the statement above (see [1, Theorem 3.1]). The following is the convergence result of the algorithm.

Theorem 1.1. *Assume for each $i \in [N]$, the constraining polynomial tuple g_i is non-singular. Let $\lambda_{i,j}(\mathbf{x})$ be the polynomial expressions of Lagrange multipliers, the \mathcal{G} be the feasible set of (1.5) with $\mathcal{K}_i = \emptyset$ for each $i \in [N]$ and \mathcal{G}^* be the set of all NEs. If the complement $\mathcal{G} \setminus \mathcal{G}^*$ is a finite set, i.e., the cardinality $\ell := |\mathcal{G} \setminus \mathcal{G}^*| < \infty$, then the proposed algorithm must terminate within at most ℓ loops, either report the nonexistence of NEs, or find one NE if it exists.*

Moreover, assume that \mathbf{x}^* is a Nash Equilibrium produced by our proposed algorithm, i.e., \mathbf{x}^* is also a minimizer of (1.5). If \mathbf{x}^* is an isolated NE (e.g., this is the case if there are finitely many NEs), we proposed a method to find more Nash equilibria. Particularly, under the assumptions of Theorem 1.1, if each NE is an isolated KKT point, and there are finitely many NEs, then the proposed method can find all of them. Please note that for a generic NEP, there are finitely many KKT points [1, Theorem A.1], hence our algorithms have finite convergence, and the proposed method for finding more Nash equilibria can find all of them.

The Moment-SOS hierarchy of semidefinite relaxations are used to solve the polynomial optimization problems in our algorithms. We refer to [1] for more details about implementing the Moment-SOS relaxation and the convergence results. Under generic assumptions, we can compute one or even all Nash equilibrium if it exists, and detect the nonexistence if there is none.

2. CONVEX GENERALIZED NASH EQUILIBRIUM PROBLEMS AND POLYNOMIAL OPTIMIZATION

The Generalized Nash Equilibrium Problem of Polynomials (GNEPP) is the game to find $\mathbf{x} \in \mathbb{R}^n$ such that each x_i is a minimizer of the i th player's optimization problem

$$(2.1) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & f_i(x_i, x_{-i}) \\ s.t. & g_{i,j}(x_i, x_{-i}) \geq 0 (j \in \mathcal{I}_i), \\ & g_{i,j}(x_i, x_{-i}) = 0 (j \in \mathcal{E}_i), \end{cases}$$

where all $f_i(x_i, x_{-i})$ and $g_{i,j}(x_i, x_{-i})$ are polynomial functions in \mathbf{x} . A solution \mathbf{x} satisfying the above is called a Generalized Nash Equilibrium (GNE). Please note in GNEPs, the i th player's feasible set depends on other player's strategies (compared with NEPs). Denote the feasible set of (2.1) by $X_i(x_{-i})$. We say a GNEPP is convex if for all $i \in [N]$ and $x_{-i} \in \text{dom}(X_i) := \{x_{-i} : X_i(x_{-i}) \neq \emptyset\}$, the $f_i(x_i, x_{-i})$ is a convex polynomial in x_i on $X_i(x_{-i})$, the $(g_{i,j}(x_i, x_{-i}))_{j \in \mathcal{I}_i}$ are concave polynomials in x_i , and $(g_{i,j}(x_i, x_{-i}))_{j \in \mathcal{E}_i}$ are affine functions in x_i . Under some regularity conditions, a feasible point $x = (x_i, x_{-i})$ is a GNE if and only if it is a KKT point, i.e., given x_{-i} , the KKT condition of the i th player's optimization is satisfied at x_i for each $i \in [N]$.

There is some existing work for finding solutions to convex GNEPs. Typically, some strong conditions are required for a guaranteed global convergence, or one may need to

solve some difficult optimization problems. Besides, there is no concrete method can detect the nonexistence of GNEs, to the best of author's knowledge. To this end, we study the convex GNEPP for a method that can detect the nonexistence of GNEs, or find one GNE if it exists.

For a typical GNEPP, the polynomial expression for Lagrange multipliers does not exist. Therefore, we introduce the rational expression of Lagrange multipliers, i.e., rational functions $\hat{\lambda}_i(x)/q_i(x)$ such that $q_i(x) \cdot \lambda = \hat{\lambda}_i(x)$ and $q_i(x) \geq 0$ hold for all feasible points. Such expressions exist generically. Besides that, we proposed another approach to express Lagrange multipliers using polynomials $\hat{\lambda}(x, \omega_i)$ in x and parameters ω_i . This is called the parametric expression of Lagrange multipliers.

For a GNEPP, assume one of these expressions exists for each player $i \in [N]$, and regularity conditions hold. Denote $\omega := (\omega_1, \dots, \omega_N)$ and $\mathbf{x} = (x, \omega)$. If x is a GNE, then it must satisfy

$$(2.2) \quad \begin{cases} \nabla_{x_i} f_i(x) - \sum_{j=1}^{m_i} \hat{\lambda}_{i,j}(\mathbf{x}) \nabla_{x_i} g_{i,j}(x) = 0 \quad (i \in [N]), \\ \hat{\lambda}_i(\mathbf{x}) \perp g_i(x), g_{i,j}(x) = 0 \quad (j \in \mathcal{E}_i, i \in [N]), \\ g_{i,j}(x) \geq 0, \hat{\lambda}_{i,j}(\mathbf{x}) \geq 0 \quad (j \in \mathcal{I}_i, i \in [N]). \end{cases}$$

In the above, $q_i(x) := 1$ if the i th player has a polynomial or partial polynomial expression of Lagrange multipliers, and $\omega_i = \emptyset$ if the polynomial or rational expression is exploited by the i th player. Denote \mathcal{G} the set of all points satisfying (2.2).

$$(2.3) \quad \begin{cases} \min_{\mathbf{x}} & [\mathbf{x}]_1^T \Theta [\mathbf{x}]_1 \\ s.t. & \mathbf{x} \in \mathcal{G}. \end{cases}$$

If (x, ω) is a feasible point of (2.3), then \mathbf{x} is a potential GNE. For a feasible point (u, w) of (2.3), assume the u represents the subvector corresponding to all player's strategies, and v is the vector of Lagrange multipliers. We have the following result.

Theorem 2.1. *For the convex GNEPP given by (2.1), suppose each Lagrange multiplier vector λ_i has a rational expression or a parametric expression.*

- (i) *If (u, w) is a feasible point of (2.3) such that $q_i(u) > 0$ for all i , then u is a GNE.*
- (ii) *Assume every GNE is a KKT point. If (2.3) is infeasible, then the GNEP has no GNEs. If Θ is positive definite and every $q_i(x) > 0$ for all feasible x of (2.3), then solving (2.3) will find a GNE if it exists.*

Therefore, one can solve the convex GNEPP by solving the polynomial optimization problem (2.3). The Moment-SOS hierarchy of semidefinite relaxations are used to solve such polynomial optimization problems. Usually, when (2.3) has a minimizer, it can be verified as a GNE. Otherwise, one may solve (2.3) with additional positivity constraints for $q_i(\mathbf{x})$, or change the positive definite matrix Θ . We refer to [2] for more details. The numerical experiments show the efficiency of our method on convex GNEPPs, some of which are difficult to be solved by other methods.

3. THE GAUSS-SEIDEL METHOD FOR GENERALIZED NASH EQUILIBRIUM PROBLEMS OF POLYNOMIALS

The Gauss-Seidel method is an iterative method for GNEPs that solves each player's optimization alternatively. It requires global minimizers for the occurring optimization problems in each loop. The following is the general framework of the Gauss-Seidel method:

Algorithm 3.1. For the GNEP of (2.1), do the following:

- Step 1. Choose a feasible starting point $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_N^{(0)})$, a positive regularization parameter $\tau^{(0)}$ and let $k := 0$.
- Step 2. If $\mathbf{x}^{(k)}$ satisfies a suitable termination criterion, stop.

Step 3. For $i = 1, \dots, N$, compute a global minimizer $x_i^{(k+1)}$ of the optimization

$$(3.1) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & f_i(x_1^{(k+1)}, \dots, x_{i-1}^{(k+1)}, x_i, x_{i+1}^{(k)}, \dots, x_N^{(k)}) + \tau^{(k)} \|x_i - x_i^{(k)}\|^2 \\ \text{s.t.} & g_{i,j}(x_i, x_{-i}) \geq 0 \ (j \in \mathcal{I}_i), \\ & g_{i,j}(x_i, x_{-i}) = 0 \ (j \in \mathcal{E}_i). \end{cases}$$

Step 4. Choose a new regularization parameter $\tau^{(k+1)} \in [0, \tau^{(k)}]$.

Step 5. Let $\mathbf{x}^{(k+1)} := (x_1^{(k+1)}, \dots, x_N^{(k+1)})$, $k := k + 1$, and go to Step 2.

In the above, the (3.1) is called a regularized optimization for the i th player. In practice, the Gauss-Seidel method performs well for solving GNEPPs. As demonstrated in [3], it can compute equilibria for many problems, some of which are nonconvex and very hard to be solved by other existing methods. However, there are several concerns when using the Gauss-Seidel method to solve GNEPPs.

- In Step 3 of Algorithm 3.1, one needs to solve the regularized optimization problem (3.1) for a global minimizer, which is difficult for a general GNEP without convexity assumption. For the GNEPPs, since the defining functions are polynomials, we use the Moment-SOS relaxations to find global minimizers of optimization problems (3.1). These relaxations can solve the polynomial optimization problems globally, even if they are nonconvex, and the global optimality can be verified. Moreover, when a limit point is obtained by Algorithm 3.1, we also use the Moment-SOS relaxations to check if it is a GNE or not.
- If the sequence by Algorithm 3.1 converges, is the limit point guaranteed to be a GNE? In [3], we give a negative answer to this question, that the [3, Example 3.5] shows a limit point may not be a solution to the GNEPP. However, we proved that if the feasible map $x_{-i} \rightarrow X_i(x_{-i})$ is inner semicontinuous, and the regularization parameter $\tau^{(k)} \rightarrow 0$, then the limit point by Algorithm 3.1 must be a GNE [3, Theorem 3.7].
- It is well known that the Gauss-Seidel method is not guaranteed to converge for all GNEPs. The generalized potential game (GPG) is a kind of GNEP such that the Gauss-Seidel method has a guaranteed convergence. We give a necessary condition for a GNEPP to be a GPG. Based on this necessary condition and Putinar's positivstellensatz, we propose a numerical certificate to check if a GNEP is GPG or not. This is the first method that can be implemented numerically for checking GPGs, to the best of the author's knowledge.

In [3], numerical experiments showing that the Gauss-Seidel method is efficient for solving many GNEPPs, even if the players' optimization problems are nonconvex. We also present some examples of using the numerical certificate to verify some GNEPs being GPGs.

4. FUTURE WORK

4.1. The nonconvex GNEPs. Consider the GNEPP defined by (2.1) without any convexity assumption on defining polynomials. In section 2, a necessary condition for a feasible point \mathbf{x} to be GNE is given by (2.2). However, such an \mathbf{x} may, or may not be, a GNE since there is no assumption on the convexity of GNEP. When a point u satisfying (2.2) but not being a GNE is obtained, we need an efficient method to find another point satisfying (2.2). Besides, to detect the nonexistence of solutions for a nonconvex GNEPP is difficult. If there is no point satisfying (2.2), then the GNEPP has no GNE. However, it is possible that the set of points satisfying (2.2) is nonempty, while there does not exist a GNE. How to exclude one or more points that is not a GNE? How to detect the nonexistence of GNE, given (2.2) has solutions? These questions are almost open, to the best of our knowledge. Moreover, an efficient method to find GNEs for nonconvex GNEPPs is still in demand.

4.2. Stochastic GNEPs. We are also interested in the Stochastic Generalized Nash Equilibrium Problems (SGNEP), i.e., the game such that for each player $i \in [N]$, the objective is an expected-value function. That is, to find $\mathbf{x} = (x_i, x_{-i})$ such that x_i minimize the following optimization for a given x_{-i} :

$$(4.1) \quad \begin{cases} \min_{x_i \in \mathbb{R}^{n_i}} & \mathbb{E}_\xi[f_i(x_i, x_{-i}, \xi)] \\ \text{s.t.} & g_{i,j}(x_i, x_{-i}) \geq 0 \ (j \in \mathcal{I}_i), \\ & g_{i,j}(x_i, x_{-i}) = 0 \ (j \in \mathcal{E}_i). \end{cases}$$

In the above, the ξ is a random vector, and the \mathbb{E} denotes the expectation of a function. Typically, the distribution of ξ is unknown. The SGNEPs are becoming increasingly important in recent years as they are widely used in game theoretic models in various of data-driven applications. However, solving the SGNEPs is difficult, and a lack of efficient method on solving SGNEPs limits the usage of these models in application.

4.3. Large scale GNEPs and applications. The NEPs and GNEPs have various applications in many different areas with increasing importance. In some applications, it is required to solve large scale NEPs and GNEPs. The methods we introduced in Section 1 and 2, though with good theoretical properties, require to solve Moment-SOS relaxations, and they are not suitable to be implemented on large scale problems. Developing methods with both theoretical and practical advantages in solving these problems is difficult, interesting, and important.

REFERENCES

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